PLX, PXA, together are not less (P.IX.) than a right angle, in either hypothesis, of right angle, or of obtuse angle; if these two angles are taken away from the sum of the given angles the then remaining angle PAD will be less than a right angle. Consequently we will be in the case of the two preceding propositions, since it is obvious that one or the other hypothesis holds, either of right angle, or of obtuse angle.

Wherefore the straights AD, and PL, or XL, meet in some point at a finite, or terminated distance on the side noted, as well under the one, as under the other mentioned hypothesis. Quod erat demonstrandum.

[To be Continued.]

THE INSCRIPTION OF REGULAR POLYGONS.

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CHAPTER IV.

[Continued from the November Number.]

We will now consider regular polygons the number of whose sides is a *composite* number. The present chapter will be devoted to the case when n is divisible by 3.

In the regular polygon of 21 *sides*, the chord $A_7=1$; and $A_3-A_6+A_9=1$, being the chords of a regular 7-gon.

But $A_1-A_2+A_3-A_4+A_5-A_6+A_7-A_8+A_9-A_{10}=1$.

Hence, $(A_1-A_8)+(-A_2+A_5)+(-A_4-A_{10})=-1$.

Now $(A_1-A_8)(-A_4-A_{10})=-[(-A_4-A_{10})+(-A_2+A_5)+2(A_3+A_9)]$.

$$(A_1-A_8)(-A_2+A_5)=-[(-A_4-A_{10})+(A_1-A_8)+2(A_3-A_6)].$$

$$(-A_2+A_5)(-A_2-A_{10})=-[(A_1-A_8)+(-A_2+A_5)+2(A_9-A_6)].$$

The sum of these 3 products$=-2[(A_1-A_8)+(-A_2+A_5)+(-A_4-A_{10})$
$$+2(A_3-A_6+A_9)]=-2.$$

Again, $(A_1-A_8)(-A_2+A_5)(-A_4-A_{10})=(A_2-A_5)[(-A_4-A_{10})$
$+(-A_2+A_5)+2(A_3+A_9)]=(A_1-A_8)+(-A_2+A_5)+(-A_4-A_{10})$
$+2(A_3-A_6+A_9)=1$; for $(A_2-A_5)(A_3+A_9)=(A_1-A_8)+(-A_2+A_5)$
$+(-A_4-A_{10})+2=1$; and $(A_2-A_5)^2=2+A_4+A_{10}-2A_3$.

$\therefore$ (A_1-A_8), $(-A_2+A_5)$ and $(-A_4-A_{10})$ are the roots of the cubic

$$x^3+x^2-2x-1=0.$$

But it was shown in Chapter I that the chords $A_3,-A_6,A_9$ of the regular 7-gon are the roots of $x^3-x^2-2x+1=0$.

Thus the two sets of roots are numerically equal but of opposite sign.

$\therefore\ A_1-A_3=A_6;\ -A_2+A_5=-A_9;\ A_4+A_{10}=A_3.$

We may write these symmetrically:

$$\begin{cases} A_1-A_6-A_8=0 \\ A_2-A_5-A_9=0 \\ A_3-A_4-A_{10}=0. \end{cases}$$

Now $(-A_1.A_6-A_1.A_8+A_6.A_8)=(A_2-A_5-A_9-3A_7)=-3.$

$A_1.A_6.A_8=A_1(A_2-A_7)=A_1+A_3-A_6-A_8=A_3.$

Hence, $A_1,-A_6,-A_8$ are the roots of $x^3-3x-A_3=0.$

Similarly, $A_2,-A_5,-A_9$ are the roots of $x^3-3x-A_6=0.$

$A_3,-A_4,\ -A_{10}$ are the roots of $x^3-3x-A_9=0.$

In the regular polygon of 27 *sides*, $A_9=1;\ A_3-A_6+A_9-A_{12}=1.$ But $A_1-A_2+A_3-A_4+A_5-A_6+\ldots\ldots+A_{13}=1.$

$\therefore\ (A_1-A_8-A_{10})+(-A_2+A_7+A_{11})+(-A_4+A_5+A_{13})=0.$

Now $(A_1-A_8-A_{10})(-A_2+A_7+A_{11})=-3(A_1-A_8-A_{10})$ $+3(A_{12}+A_6-A_3)=-3(A_1-A_8-A_{10})$. Hence, either $(-A_2+A_7+A_{11})$ $=-3$, or $(A_1-A_8-A_{10})=0$. Again, $(A_1-A_8-A_{10})(-A_4+A_5+A_{13})$ $=-3(-A_4+A_5+A_{13})$. Hence, either $(A_1-A_8-A_{10})=-3$, or $(-A_4+A_5+A_{13})=0$. Lastly, $(-A_2+A_7+A_{11})(-A_4+A_5+A_{13})$ $=-3(-A_2+A_7+A_{11})$. Hence, either $(-A_4+A_5+A_{13})=-3$, or

$(-A_2+A_7+A_{11})=0.$

But $(A_1-A_8-A_{10})+(-A_2+A_7+A_{11})+(-A_4+A_5+A_{13})=0.$

Hence each term can not equal -3, but must be 0.

$$\therefore\ \begin{cases} A_1-A_8-A_{10}=0 \\ A_2-A_7-A_{11}=0 \\ A_3-A_6-A_{12}=0 \\ A_4-A_5-A_{13}=0. \end{cases}$$

Now $(-A_1.A_8-A_1.A_{10}+A_8.A_{10})=-[3+(-A_2+A_7+A_{11})]=-3.$

$A_1.A_8.A_{10}=A_1(A_2-A_9)=A_3+(A_1-A_8-A_{10})=A_3.$

Hence, $A_1,-A_8,-A_{10}$ are the roots of $x^3-3x-A_3=0.$

Similarly, $A_2,-A_7,-A_{11}$ are the roots of $x^3-3x-A_6=0.$

$A_4,-A_5,-A_{13}$ are the roots of $x^3-3x-A_{12}=0.$

By induction we derive that for a regular polygon of $n=3m$ sides:

$$A_1-A_{m-1}-A_{m+1}=0$$

$$A_2-A_{m-2}-A_{m+2}=0$$

Generally, $A_s-A_{m-s}-A_{m+s}=0.$

To prove the general formula, throw it into its trigonometric form,

$$\cos\frac{s\pi}{3m}-\cos\frac{(m-s)\pi}{3m}-\cos\frac{(m+s)\pi}{3m}=0,$$ which follows since

$$\cos\frac{(m-s)\pi}{3m}+\cos\frac{(m+s)\pi}{3m}=2\cos\frac{\pi}{3}.\cos\frac{s\pi}{3m}=\cos\frac{s\pi}{3m}.$$

Again, $(-A_1.A_{m-1}-A_1.A_{m+1}+A_{m-1}.A_{m+1})=(A_2-A_{m-2}-A_{m+2})-3A_m$
$=-3$, since A_m is the unit radius.

$$A_1.A_{m-1}.A_{m+1}=A_1(A_2-A_m)=A_1(A_2-1)=A_1+A_3-A_1=A_3.$$

$\therefore$ $A_1, -A_{m-1}, -A_{m+1}$ are the roots of $x^3-3x-A_3=0$.

Similarly, A_2, $-A_{m-2}, -A_{m+2}$ are the roots of $x^3-3x-A_6=0$.

Generally, A_s, $-A_{m-s}$, $-A_{m+s}$ are the roots of $x^3-3x-A_{3s}=0$,

where s is and integer $\leqq \frac{m-1}{2}$; and $A_3, -A_6, A_9, \ldots (-1)^{s-1}A_{3s}, \ldots\ldots$ are the $\frac{m-1}{2}$ roots of the general equation (4) of Chapter II. for the regular m-gon.

If m is prime to 3,i.e., if n contains the factor 3 in but the first degree, one chord out of every group of three chords given above is a root of this equation. For, m being prime to 3, one and only one of the subscripts s, $m-s$, $m+s$ is disisible by 3, as is seen by writing them respectively $3m-s$, $m-s$, $2m-s$.

The remaining two chords in the group will be roots of a quadratic whose coefficients are linear functions of the roots of equation (4).

Thus, if $m+s$ be divisible by 3, $m-2s$ will also.

Then $A_s-A_{m-s}=A_{m+s}$. $A_s.A_{m-s}=A_m+A_{m-2s}=1+A_{m-2s}$.

Hence, A_s and $-A_{m-s}$ are the roots of $x^2-A_{m+s}x-(1+A_{m-2s})=0$, in which A_{m+s} and A_{m-2s} are roots of the equation (4) for the regular m-gon. Hence, if the $\frac{m-1}{2}$ chords of the regular m-gon be found, we can find all the chords of the regular 3 m-gon by solving a series of quadratics.

However, if m is divisible by 3, i.e.,if n contains the factor 3^2, the three chords in each of the above groups must either all or none be roots of (4); for the subscripts s, $m-s$, $m+s$, are then either all or none divisible by 3 according as s is divisible by 3 or not. Hence we can neither lower the cubics nor avoid them.

The regular 3m-gon depends, therefore, for inscription upon the same equations as does the regular m-gon, if m be prime to 3; *but depends upon one or more cubics in addition to the former equations, if m contains the factor* 3.

ERRATA in Chapter III p 376 line 1, *for* form *read* degree p 377 lines 1 and 5, extend radical sign over 2 $(17\pm\sqrt{17})$, line 14, *for* $(-A_1-2A_2+\ldots)$ *read* $-(A_1-2A_2+\ldots)$, line 18, seperate the two products thus] ; (, line 21, *for* $2b$ read $2B$.

[To be Continued.]